

THE C. NEUMANN PROBLEM AS A COMPLETELY INTEGRABLE SYSTEM ON AN ADJOINT ORBIT

BY

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ABSTRACT. It is shown by purely Lie algebraic methods that the C. Neumann problem—the motion of a material point on a sphere under the influence of a quadratic potential—is a completely integrable system of Euler-Poisson equations on a minimal-dimensional orbit of a semidirect product of Lie algebras.

1. The C. Neumann problem. The motion of a point on the sphere S^{n-1} under the influence of a quadratic potential $U(\mathbf{x}) = \frac{1}{2} A \mathbf{x} \cdot \mathbf{x}$, $\mathbf{x} \in \mathbf{R}^n$, $A = \text{diag}(a_1, \dots, a_n)$ is a completely integrable Hamiltonian system. For $n = 3$ this has been shown by C. Neumann in 1859 [12] and for arbitrary n by K. Uhlenbeck [16], R. Devaney [3], J. Moser [10], [11], M. Adler, and P. van Moerbeke [2]. In this paper we show how this problem fits naturally in the framework of Euler-Poisson equations [4], [5], [14], [17] proving that the C. Neumann problem is a Hamiltonian system on a minimal-dimensional adjoint orbit in a semidirect product of Lie algebras. Thus its complete integrability will follow entirely from Lie algebraic considerations.

The equations of motion are

$$\ddot{x}_i = -a_i x_i + \lambda x_i, \quad i = 1, \dots, n, \quad (1.1)$$

where the Lagrange multiplier $\lambda = A \mathbf{x} \cdot \mathbf{x} - \|\mathbf{x}\|^2$ is chosen such that $\mathbf{x} \in S^{n-1}$ during the motion. Set $\dot{\mathbf{x}} = \mathbf{y}$ and get the equivalent system to (1.1)

$$\dot{x}_i = y_i, \quad \dot{y}_i = -a_i x_i + (A \mathbf{x} \cdot \mathbf{x} - \|\mathbf{y}\|^2) x_i, \quad \|\mathbf{x}\| = 1, \quad \mathbf{x} \cdot \mathbf{y} = 0. \quad (1.2)$$

The following crucial remark that motivated the present investigation is due to K. Uhlenbeck and can be verified without any difficulties.

LEMMA 1.1. Put $X = (x_i x_j)$, $P = (y_i x_j - x_i y_j)$. System (1.2) is equivalent to

$$\dot{X} = [P, X], \quad \dot{P} = [X, A], \quad \|\mathbf{x}\| = 1, \quad \mathbf{x} \cdot \mathbf{y} = 0. \quad (1.3)$$

Remark that if one replaces X and A by $X - \text{Id}/n$ and $A - (\text{Tr}(A))\text{Id}/n$ respectively, where Id is the $n \times n$ identity matrix, equations (1.3) remain unchanged. From now on we shall assume that in (1.3) this change has been made so that $X, P, A \in \mathfrak{sl}(n)$. The next section gives a Lie algebraic interpretation to these equations.

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2. The equations of motion as a Hamiltonian system on an adjoint orbit. We start by recalling a few facts about the ad-semidirect product $\mathfrak{G}_{\text{ad}} \times \overline{\mathfrak{G}}$ of a semisimple Lie algebra $\mathfrak{G}$ with the abelian Lie algebra $\overline{\mathfrak{G}}$ having underlying vector space $\mathfrak{G}$. If $(\xi_1, \eta_1), (\xi_2, \eta_2) \in \mathfrak{G}_{\text{ad}} \times \overline{\mathfrak{G}}$, their bracket is defined by

$$[(\xi_1, \eta_1), (\xi_2, \eta_2)] = ([\xi_1, \xi_2], [\xi_1, \eta_2] - [\xi_2, \eta_1]). \quad (2.1)$$

If κ denotes a bilinear, symmetric, nondegenerate, bi-invariant, two-form on $\mathfrak{G}$, the form κ_s , called the semidirect product of κ with itself and defined by

$$\kappa_s((\xi_1, \eta_1), (\xi_2, \eta_2)) = \kappa(\xi_1, \eta_2) + \kappa(\xi_2, \eta_1) \quad (2.2)$$

is a bilinear, symmetric, nondegenerate, bi-invariant, two-form on $\mathfrak{G}_{\text{ad}} \times \overline{\mathfrak{G}}$.

Let G be a Lie group with Lie algebra $\mathfrak{G}$. The Ad-semidirect product $G_{\text{Ad}} \times \overline{\mathfrak{G}}$ is a Lie group with underlying manifold $G \times \mathfrak{G}$ and composition law

$$(g_1, \xi_1)(g_2, \xi_2) = (g_1 g_2, \xi_1 + \text{Ad}_{g_1} \xi_2). \quad (2.3)$$

Note that the identity element is $(e, 0)$ and the inverse $(g, \xi)^{-1} = (g^{-1}, -\text{Ad}_{g^{-1}} \xi)$. The Lie algebra of $G_{\text{Ad}} \times \overline{\mathfrak{G}}$ is $\mathfrak{G}_{\text{ad}} \times \overline{\mathfrak{G}}$. The adjoint action of the Lie group $G_{\text{Ad}} \times \overline{\mathfrak{G}}$ on $\mathfrak{G}_{\text{ad}} \times \overline{\mathfrak{G}}$ is given by

$$\text{Ad}_{(g, \theta)}(\xi, \eta) = (\text{Ad}_g \xi, \text{Ad}_g \eta + [\theta, \text{Ad}_g \xi]). \quad (2.4)$$

In the considerations that follow, the orbit symplectic structure plays a central role (see Abraham and Marsden [1] and Ratiu [14] for proofs). If a Lie algebra $\mathfrak{G}$ has a bilinear, symmetric, nondegenerate, bi-invariant two-form κ ,

$$\omega_\xi(\text{Ad}_g \xi)([\eta, \text{Ad}_g \xi], [\zeta, \text{Ad}_g \xi]) = -\kappa([\eta, \zeta], \text{Ad}_g \xi) \quad (2.5)$$

for $\xi, \eta, \zeta \in \mathfrak{G}$, $g \in G$, defines the canonical symplectic structure on the adjoint orbit $G \cdot \xi$ through ξ . If $E, E': \mathfrak{G} \rightarrow \mathbb{R}$, the Hamiltonian vector field of $E|_{G \cdot \xi}$ is given by

$$X_{E|_{G \cdot \xi}}(\text{Ad}_g \xi) = -[(\text{grad } E)(\text{Ad}_g \xi), \text{Ad}_g \xi] \quad (2.6)$$

and the Poisson bracket of $E|_{G \cdot \xi}, E'|_{G \cdot \xi}$ is

$$\{E|_{G \cdot \xi}, E'|_{G \cdot \xi}\}(\text{Ad}_g \xi) = -\kappa([(\text{grad } E)(\text{Ad}_g \xi), (\text{grad } E')(\text{Ad}_g \xi)], \text{Ad}_g \xi) \quad (2.7)$$

where grad denotes the gradient with respect to κ .

For the semidirect product these formulas become

$$\begin{aligned} \omega(\xi, \eta)([(\xi, \eta), (\xi_1, \xi_1')], [(\xi, \eta), (\xi_2, \xi_2')]) \\ = -\kappa_s((\xi, \eta), [(\xi_1, \xi_1'), (\xi_2, \xi_2')]), \end{aligned} \quad (2.8)$$

$$X_E(\xi, \eta) = -([(\text{grad}_2 E)(\xi, \eta), \xi], [(\text{grad}_2 E)(\xi, \eta), \eta] + [(\text{grad}_1 E)(\xi, \eta), \xi]), \quad (2.9)$$

$$\begin{aligned} \{E, E'\}(\xi, \eta) = & -\kappa(\xi, [(\text{grad}_2 E)(\xi, \eta), (\text{grad}_1 E')(\xi, \eta)]) \\ & -\kappa(\xi, [(\text{grad}_1 E)(\xi, \eta), (\text{grad}_2 E')(\xi, \eta)]) \\ & -\kappa(\eta, [(\text{grad}_2 E)(\xi, \eta), (\text{grad}_2 E')(\xi, \eta)]), \end{aligned} \quad (2.10)$$

where $(\text{grad}_1, \text{grad}_2)$ denotes the usual gradient with respect to $\kappa \times \kappa$; note that the gradient with respect to κ_s is $(\text{grad}_2, \text{grad}_1)$.

Assume that $\mathfrak{G} = \mathfrak{R} \oplus \mathfrak{N}$, where $\mathfrak{R}$ is a vector subspace and $\mathfrak{N}$ a Lie subalgebra of $\mathfrak{G}$, $\mathfrak{N}$ having N as underlying closed Lie subgroup of G . Denote by $\Pi_{\mathfrak{R}}, \Pi_{\mathfrak{N}}$ the projections of $\mathfrak{G}$ on $\mathfrak{R}$ and $\mathfrak{N}$ respectively. Then $\mathfrak{G}^* = \mathfrak{R}^* \oplus \mathfrak{N}^*$. The nondegeneracy of κ on $\mathfrak{G}$ defines the isomorphisms $\mathfrak{N}^\perp \cong \mathfrak{R}^*$, $\mathfrak{R}^\perp \cong \mathfrak{N}^*$ and thus the coadjoint action of N on $\mathfrak{N}^*$ induces an action of N on $\mathfrak{R}^\perp$ given by

$$N \times \mathfrak{R}^\perp \ni (n, \xi) \mapsto \Pi_{\mathfrak{R}^\perp} \text{Ad}_n \xi \in \mathfrak{R}^\perp \quad (2.11)$$

where $\Pi_{\mathfrak{R}^\perp}: \mathfrak{G} \rightarrow \mathfrak{R}^\perp$ denotes the canonical projection defined by the direct sum decomposition $\mathfrak{G} = \mathfrak{R}^\perp \oplus \mathfrak{N}^\perp$. Thus the orbit $N \cdot \xi$ equals

$$N \cdot \xi = \{ \Pi_{\mathfrak{R}^\perp}(\text{Ad}_n \xi) | n \in N \} \subseteq \mathfrak{R}^\perp, \quad \xi \in \mathfrak{R}^\perp, \quad (2.12)$$

whose tangent space at $\bar{\xi} \in N \cdot \xi$ is

$$T_{\bar{\xi}}(N \cdot \xi) = \{ \Pi_{\mathfrak{R}^\perp}[\bar{\xi}, \eta] | \eta \in \mathfrak{N} \} \subseteq \mathfrak{R}^\perp. \quad (2.13)$$

The symplectic structure on $N \cdot \xi$ equals, by (2.5),

$$\omega_{\bar{\xi}}(\bar{\xi})(\Pi_{\mathfrak{R}^\perp}[\eta, \bar{\xi}], \Pi_{\mathfrak{R}^\perp}[\zeta, \bar{\xi}]) = -\kappa([\eta, \zeta], \bar{\xi}), \quad \bar{\xi} \in N \cdot \xi \subseteq \mathfrak{R}^\perp, \quad (2.14)$$

and the Hamiltonian vector field defined by $E|N \cdot \xi$, $E: \mathfrak{G} \rightarrow \mathbb{R}$, is, by (2.6),

$$X_{E|N \cdot \xi}(\bar{\xi}) = -\Pi_{\mathfrak{R}^\perp}[\Pi_{\mathfrak{N}}(\text{grad } E)(\bar{\xi}), \bar{\xi}], \quad \bar{\xi} \in N \cdot \xi \subseteq \mathfrak{R}^\perp. \quad (2.15)$$

Finally the Poisson bracket of $E|G \cdot \xi$, $E'|G \cdot \xi$ is given by (2.7),

$$\{E|G \cdot \xi, E'|G \cdot \xi\}(\bar{\xi}) = -\kappa([\Pi_{\mathfrak{N}}(\text{grad } E)(\bar{\xi}), \Pi_{\mathfrak{N}}(\text{grad } E')(\bar{\xi})], \bar{\xi}), \quad (2.16)$$

for $\bar{\xi} \in N \cdot \xi \subseteq \mathfrak{R}^\perp$. All previous considerations naturally live on the duals but this is the form we shall use for the C. Neumann problem; see Ratiu [14] for a parallel description on duals.

We shall apply all previous results to a specific Lie algebra. Let $\mathfrak{G} = \mathfrak{sl}(n)_{\text{ad}} \times \overline{\mathfrak{sl}(n)}$, $G = \text{Sl}(n)_{\text{Ad}} \times \overline{\text{sl}(n)}$, $\mathfrak{N} = \mathfrak{so}(n) \times \text{sym}$, $N = \text{SO}(n) \times \text{sym}$, $\mathfrak{R} = \text{sym} \times \mathfrak{so}(n)$, where $\text{sym} \subset \mathfrak{sl}(n)$ denotes the vector space of all symmetric matrices. Clearly $\mathfrak{N}$ is a Lie subalgebra and $\mathfrak{R}$ a vector subspace of $\mathfrak{G}$, N a Lie subgroup of G with Lie algebra $\mathfrak{N}$. Thus by our general considerations N acts on $\mathfrak{R}^\perp$. It is easy to check that with respect to κ , where $\kappa(A, B) = -\frac{1}{2}\text{Tr}(AB)$, $\mathfrak{R}^\perp = \mathfrak{R}$, $\mathfrak{N}^\perp = \mathfrak{N}$. In what follows we shall determine explicitly a particular N -orbit; note first that in the case above $\Pi_{\mathfrak{R}^\perp}$ in formula (2.11) is not necessary, i.e. the action of N on $\mathfrak{R}^\perp$ is given by (2.4).

If $y, z \in \mathbb{R}^n$, denote by $y \otimes z$ the matrix having entries $y_i z_j$ and remark that if $g \in \text{SO}(n)$, $g(y \otimes z)g^{-1} = (gy) \otimes (gz)$. Let $z = (1, \dots, 1)/\sqrt{n}$ and take $(z \otimes z - \text{Id}/n, 0) \in \mathfrak{R}^\perp$. Let $g \in \text{SO}(n)$ be arbitrary and denote $x = gz$. Then $\|x\| = \|z\| = 1$ and $g(z \otimes z - \text{Id}/n)g^{-1} = x \otimes x - \text{Id}/n$ which is a matrix X having all off-diagonal entries equal to $x_i x_j$ and diagonal entries $x_i^2 - 1/n$. Thus the first component of the N -orbit through $(z \otimes z - \text{Id}/n, 0)$ is the matrix X occurring in Lemma 1.1. We compute the second component. If $\theta \in \text{sym}$, $X_{ij} = x_i x_j$, $X_{ii} = x_i^2 - 1/n$, then

$$[\theta, X]_{ij} = \left(\sum_{k=1}^n \theta_{ik} x_k - x_i C(x, \theta) \right) x_j - x_i \left(\sum_{k=1}^n \theta_{jk} x_k - x_j C(x, \theta) \right),$$

where $C(\mathbf{x}, \theta) = \sum_{k,i} x_i x_k \theta_{ik}$. Put $y_i = \sum_{k=1}^n (\theta_{ik} x_k - x_i C(\mathbf{x}, \theta))$ and remark that if $\mathbf{y} = (y_1, \dots, y_n)$, $\mathbf{x} \cdot \mathbf{y} = 0$ since $\|\mathbf{x}\| = 1$. Thus the second component of the N -orbit consists of matrices $P \in so(n)$, $P_{ij} = y_i x_j - x_j y_i$, $\mathbf{x} \cdot \mathbf{y} = 0$. We showed hence that this N -orbit consists of pairs $(X, P) \in sym \times so(n)$ with X, P defined as in Lemma 1.1.

Remark that the correspondence $(X, P) = \lambda(\mathbf{x}, \mathbf{y})$ defines a diffeomorphism of this orbit onto the tangent bundle TS^{n-1} of the unit sphere S^{n-1} in $\mathbf{R}^n$. A tangent vector at (X, P) to this orbit is $[(X, P), (\xi, \eta)]$ for $(\xi, \eta) \in so(n) \times sym$ and is of the form $(V, W) \in sym \times so(n)$, where $V_{ij} = v_j x_i + x_j v_i$, $v_i = \sum_{k=1}^n x_k \xi_{ki}$, $W_{ij} = w_i x_j - x_i w_j + y_i v_j - y_j v_i$, $w_i = \sum_{k=1}^n (y_k \xi_{ki} - x_k \eta_{ki}) + x_i \sum_{k,i} x_i x_k \eta_{ik}$ as a short calculation shows. Thus the tangent map of λ is given by $(V, W) \mapsto (\mathbf{v}, \mathbf{w})$, where $\mathbf{v} = (v_1, \dots, v_n)$, $\mathbf{w} = (w_1, \dots, w_n)$.

TS^{n-1} has a natural symplectic structure induced by the canonical symplectic form $-\sum_{i=1}^n dx_i \wedge dy_i$ of $\mathbf{R}^{2n}$. By (2.8) the canonical symplectic structure ω on the orbit is given by

$$\omega(X, P)([(X, P), (\xi^1, \eta^1)], [(X, P), (\xi^2, \eta^2)]) = \kappa_s([(X, P), (\xi^2, \eta^2)], [(X, P), (\xi^1, \eta^1)]).$$

Let $V^i, W^i, \mathbf{v}^i, \mathbf{w}^i$ be defined by $\xi^i, \eta^i, i = 1, 2$. We have by bi-invariance of κ_s , antisymmetry of ξ^2 , symmetry of η^1 , and by (2.8), (2.14)

$$\begin{aligned} (\lambda_* \omega)(\mathbf{x}, \mathbf{y})((\mathbf{v}^1, \mathbf{w}^1), (\mathbf{v}^2, \mathbf{w}^2)) &= \omega(X, P)((V^1, W^1), (V^2, W^2)) \\ &= -\kappa_s((\xi^2, \eta^2), (V^1, W^1)) \\ &= \frac{1}{2} \text{Tr}(\xi^2 W^1) + \frac{1}{2} \text{Tr}(\eta^2 V^1) \\ &= \sum_{k=1}^n \left(w_k^1 \sum_{i=1}^n x_i \xi_{ik}^2 \right) - \sum_{k=1}^n \left(v_k^1 \sum_{i=1}^n y_i \xi_{ik}^2 \right) + \sum_{k=1}^n \left(v_k^1 \sum_{i=1}^n x_i \eta_{ik}^2 \right) \\ &= - \sum_{k=1}^n (v_k^1 w_k^2 - w_k^1 v_k^2) \\ &= \left(- \sum_{k=1}^n dx_k \wedge dy_k \right)((\mathbf{v}^1, \mathbf{w}^1), (\mathbf{v}^2, \mathbf{w}^2)). \end{aligned}$$

This shows that λ is a symplectic diffeomorphism:

$$\lambda_* \omega = \left(- \sum_{i=1}^n dx_i \wedge dy_i \right) |_{TS^{n-1}}.$$

Let $L: sl(n) \rightarrow sl(n)$ be given by $L(\xi) = -\xi$. L is clearly a κ -symmetric isomorphism. The following Euler-Poisson Hamiltonian (see [4], [5], [15], [17] for motivations) $E(\xi, \eta) = \frac{1}{2} \kappa(\eta, L(\eta)) + \kappa(A, \xi)$ for $A \in sl(n)$ a fixed diagonal matrix, induces a Hamiltonian vector field on the N -orbit through $(\mathbf{z} \otimes \mathbf{z} - \text{Id}/n, 0)$ given by (2.15), $(X, P) \mapsto ([P, X], [X, A])$, i.e. we get equations (1.3). Hence we proved the following.

THEOREM 2.1. *The $N = so(n) \times \text{sym}$ -orbit through $(z \otimes z - \text{Id}/n, 0)$ in $\mathbb{R}^1 = \mathbb{R} = \text{sym} \times so(n)$, $z = (1, \dots, 1)/\sqrt{n}$, consists of all pairs (X, P) , $X_{ij} = x_i x_j$ for $i \neq j$, $X_{ii} = x_i^2 - 1/n$, $P_{ij} = y_i x_j - x_i y_j$, $\|x\| = 1$, $x \cdot y = 0$. With the Kirillov-Kostant-Souriau symplectic structure this $(2n - 2)$ -dimensional orbit is symplectically diffeomorphic via $(X, P) \mapsto (x, y)$ to TS^{n-1} with the symplectic structure induced from $\mathbb{R}^{2n}$ by $-\sum_{i=1}^n dx_i \wedge dy_i$. The Hamiltonian $E(X, P) = -\frac{1}{2}\kappa(P, P) + \kappa(A, X)$ defines on this orbit the equations of motion of the C. Neumann problem*

$$\dot{X} = [P, X], \quad \dot{P} = [X, A], \quad \|x\| = 1, \quad x \cdot y = 0. \quad (2.17)$$

REMARK. M. Adler and P. van Moerbeke [2] have independently observed that (2.17) is a Hamiltonian system in a semidirect product.

3. The complete set of integrals and their involution.

LEMMA 3.1. *The equations $\dot{X} = [P, X]$, $\dot{P} = [X, A]$, $\|x\| = 1$, $x \cdot y = 0$ are equivalent to*

$$(-X + P\lambda + A\lambda^2)' = [-X + P\lambda + A\lambda^2, -P - A\lambda] \quad (3.1)$$

for any parameter λ .

The proof is a straightforward verification. It follows that the functions $(1/2(k+1))\text{Tr}(-X + P\lambda + A\lambda^2)^{k+1}$ are conserved on the flow of (3.1). If $t \mapsto (X(t), P(t))$ denotes the flow of (2.17), then $t \mapsto -X(t) + P(t)\lambda + A\lambda^2$ is the flow of (3.1) and we conclude that the coefficients of λ in the expansion of $(1/2(k+1))\text{Tr}(-X + P\lambda + A\lambda^2)^{k+1}$ are conserved along the flow of (2.17). Let $f_k(X, P)$ be the coefficient of λ^{2k} in this expansion for $k = 1, \dots, n-1$. We shall prove in this section that all f_k are in involution. The method of the proof follows Ratiu [13], [15] closely.

THEOREM 3.2. *Let $\mathfrak{G}$ be a Lie algebra with a bilinear, symmetric, nondegenerate, bi-invariant, two-form κ . Let $f, g: \mathfrak{G} \rightarrow \mathbb{R}$ satisfy $[(\text{grad } f)(\xi), \xi] = 0$, $[(\text{grad } g)(\xi), \xi] = 0$ for all $\xi \in \mathfrak{G}$. Denote $f_a(\xi, \eta) = f(\xi + a\eta + a^2\epsilon)$, $g_b(\xi, \eta) = g(\xi + b\eta + b^2\epsilon)$ for $\epsilon \in \mathfrak{G}$ fixed and a, b arbitrary parameters. Then f_a, g_b Poisson commute in the bracket of $\mathfrak{G}_{\text{ad}} \times \mathfrak{G}$ defined by its symplectic decomposition in adjoint orbits.*

PROOF. Clearly $(\text{grad}_1 f_a)(\xi, \eta) = (\text{grad } f)(\xi + a\eta + a^2\epsilon)$, $(\text{grad}_2 f_a)(\xi, \eta) = a(\text{grad } f)(\xi + a\eta + a^2\epsilon)$ and similarly for g_b . By (2.10)

$$\begin{aligned} \{f_a, g_b\}(\xi, \eta) &= -\kappa([(\text{grad}_1 f_a)(\xi, \eta), (\text{grad}_2 g_b)(\xi, \eta)], \xi) \\ &\quad - \kappa([(\text{grad}_2 f_a)(\xi, \eta), (\text{grad}_1 g_b)(\xi, \eta)], \xi) \\ &\quad - \kappa([(\text{grad}_2 f_a)(\xi, \eta), (\text{grad}_2 g_b)(\xi, \eta)], \eta) \\ &= -\kappa((a+b)\xi + ab\eta, [(\text{grad } f)(\xi + a\eta + a^2\epsilon), \\ &\quad (\text{grad } g)(\xi + b\eta + b^2\epsilon)]) \\ &= (b^2/(a-b))\kappa([\xi + a\eta + a^2\epsilon, (\text{grad } f)(\xi + a\eta + a^2\epsilon)], \\ &\quad (\text{grad } g)(\xi + b\eta + b^2\epsilon)) \\ &\quad + (a^2/(a-b))\kappa([\xi + b\eta + b^2\epsilon, (\text{grad } g)(\xi + b\eta + b^2\epsilon)], \\ &\quad (\text{grad } f)(\xi + a\eta + a^2\epsilon)) = 0 \end{aligned}$$

by hypothesis. By continuity $\{f_a, g_b\} = 0$ holds also for $a = b$. $\square$

REMARK. The condition $[(\text{grad } f)(\xi), \xi] = 0$ for all $\xi \in \mathfrak{G}$ is the infinitesimal version of Ad-invariance of f as an easy computation shows.

THEOREM 3.3. Let G be a Lie group, N a closed subgroup, with Lie algebras $\mathfrak{G}$ and $\mathfrak{N}$ respectively. Assume $\mathfrak{G} = \mathfrak{R} \oplus \mathfrak{N}$, $\mathfrak{R}$ a vector subspace, $[\mathfrak{R}, \mathfrak{N}] \subseteq \mathfrak{R}$, and that $\mathfrak{G}$ has a bilinear, symmetric, nondegenerate, bi-invariant, two-form κ . Assume that $f, g: \mathfrak{G} \rightarrow \mathbf{R}$ Poisson commute on $\mathfrak{G}$, i.e.

$$\kappa([\text{grad } f](\xi), [\text{grad } g](\xi)), \xi) = 0,$$

for all $\xi \in \mathfrak{G}$. If either

(1) $\mathfrak{R}$ is a Lie subalgebra, or

(2) $\Pi_{\mathfrak{N}}[\Pi_{\mathfrak{R}}(\text{grad } f)(\eta), \Pi_{\mathfrak{R}}(\text{grad } g)(\eta)] = 0$ for all $\eta \in \mathfrak{R}^\perp$,

then on any N -orbit in $\mathfrak{R}^\perp$, the functions f, g Poisson commute.

PROOF. Let $\eta \in \mathfrak{R}^\perp$. By hypothesis and (2.16) we get

$$\begin{aligned} 0 &= -\kappa([\text{grad } f](\eta), [\text{grad } g](\eta)), \eta) \\ &= \{f|N \cdot \eta, g|N \cdot \eta\}(\eta) - \kappa(\Pi_{\mathfrak{N}}[\Pi_{\mathfrak{R}}(\text{grad } f)(\eta), \Pi_{\mathfrak{R}}(\text{grad } g)(\eta)], \eta). \end{aligned}$$

The second term vanishes in either hypothesis 1 or 2. $\square$

REMARK. Both theorems have identical versions on duals and κ is not needed there.

These two general theorems prove the involution of the functions f_k in the following way. In Theorem 3.2 take $\mathfrak{G} = \mathfrak{sl}(n)$ and let

$$\phi_k(\xi, \eta) = (1/2(k+1)) \text{Tr}(-\xi + \eta \lambda_{k+1} + A \lambda_{k+1}^2)^{k+1}.$$

Then $\{\phi_k, \phi_l\} = 0$ on $\mathfrak{sl}(n)_{\text{ad}} \times \overline{\mathfrak{sl}(n)}$ for any parameters $\lambda_{k+1}, \lambda_{l+1}$, i.e. ϕ_{k+1} is constant on the flow defined by ϕ_{l+1} no matter what $\lambda_{k+1}, \lambda_{l+1}$ are, i.e. the coefficients of λ_{k+1} in ϕ_{k+1} are constant on the flow defined by ϕ_{l+1} for all λ_{l+1} . Hence $\{f_k, \phi_l\} = 0$ for all λ_l and thus $\{f_k, f_l\} = 0$ for any k, l . In Theorem 3.3 take $\mathfrak{G} = \mathfrak{sl}(n)_{\text{ad}} \times \overline{\mathfrak{sl}(n)}$, $\mathfrak{R} = \text{sym} \times \mathfrak{so}(n)$, $\mathfrak{N} = \mathfrak{so}(n) \times \text{sym}$ and remark that $[\mathfrak{R}, \mathfrak{N}] \subseteq \mathfrak{R}$. f_k, f_l Poisson commute on $\mathfrak{G}$ by what we just proved, so in order to conclude that they Poisson commute on the N -orbit through $(z \otimes z - \text{Id}/n, 0)$ we have to check condition (2) of Theorem 3.3 for $\eta = (X, P) \in \mathfrak{R}^\perp = \mathfrak{R}$. An easy computation shows that

$$f_k(X, P) = \frac{1}{2(k+1)} \text{Tr} \left[- \sum_{i=0}^k A^i X A^{k-i} + \sum_{\substack{i+j+l=k-1 \\ i, j, l \geq 0}} A^i P A^j P A^l \right] \quad (3.2)$$

so that the gradient of f_k with respect to κ_s is

$$(\text{grad } f_k)(X, P) = \left(- \sum_{i=0}^{k-1} A^i P A^{k-1-i}, A^k \right) \in \mathfrak{so}(n) \times \text{sym} = \mathfrak{N} \quad (3.3)$$

and hence $\Pi_{\mathfrak{R}}(\text{grad } f_k)(X, P) = 0$.

THEOREM 3.4. The functions f_k , $k = 1, \dots, n-1$, are constants of the motion in involution for the C. Neumann problem. $f_1 = -E$, $E = \text{energy function}$.

REMARK. Equation (3.1) is Hamiltonian in the Kac-Moody extension of $sl(n)$; see Adler and van Moerbeke [2].

4. Independence. Throughout this section we assume that $A = \text{diag}(a_1, \dots, a_n)$ has all entries distinct.

Let ${}^cV = \text{span}\{X_{f_k}(X, P) | k = 1, \dots, n-1\}$. We have to show that generically $\dim({}^cV) = n-1$.

Denote by U_{ki} the coefficient of λ^i in the expansion of $(-X + P\lambda + A\lambda^2)^k$. From (3.3) it follows that $(\text{grad } f_k)(X, P) = (-U_{k,2k-1}, U_{k,2k})$, so that ${}^cV = \text{ad}_{(X,P)}\mathcal{Q}_0$, where $\mathcal{Q}_0 = \text{span}\{(-U_{k,2k-1}, U_{k,2k}) | k = 1, \dots, n-1\}$. Since $U_{k,2k} = A^k$ and A has all entries distinct we conclude that $\{A^k | k = 1, \dots, n-1\}$ are linearly independent in $sl(n)$ and thus $\dim(\mathcal{Q}_0) = n-1$; in particular $\dim({}^cV) \leq \dim(\mathcal{Q}_0) = n-1$ which was already obvious from the definition of cV .

Let $\mathcal{Q}_j = \text{span}\{(-U_{k,2k-1-2j}, U_{k,2k-2j}) | k = j, \dots, n-1\}$ where we make the convention that any U_{ki} with $i < 0$ is identical zero; thus $\dim(\mathcal{Q}_j) \leq n-j$, $j = 1, \dots, n-1$. Denote ${}^cV_j = \text{ad}_{(X,P)}\mathcal{Q}_j$, $j = 0, 1, \dots, n-1$, so that ${}^cV = {}^cV_0$.

LEMMA 4.1. *The linear map $f_{A,P}: sl(n) \times sl(n) \rightarrow sl(n) \times sl(n)$ defined by*

$$f_{A,P}(\xi, \eta) = ([\eta, P] - [\xi, A], [A, \eta])$$

is injective on all $\mathcal{Q}_j$, $j = 1, \dots, n-1$, for generic (X, P) .

This is a direct, but somewhat lengthy verification (see [15] for a more complicated similar proof).

LEMMA 4.2. *The following relations hold for any $k = 1, \dots, n-1$:*

$$- [U_{k,2k-j}, X] + [U_{k,2k-j-1}, P] + [U_{k,2k-j-2}, A] = 0.$$

This is obvious if one notes that the expression above is the coefficient of λ^j in the expansion of $[(-X + P\lambda + A\lambda^2)^k, -X + P\lambda + A\lambda^2] \equiv 0$.

We have thus by the two prior lemmas

$$\begin{aligned} \text{ad}_{(X,P)}(-U_{k,2k-1-2j}, U_{k,2k-2j}) \\ &= ([U_{k,2k-1-2j}, X], [X, U_{k,2k-2j}] + [U_{k,2k-1-2j}, P]) \\ &= ([U_{k,2k-2-2j}, P] + [U_{k,2k-3-2j}, A], [A, U_{k,2k-2j-2}]) \\ &= f_{A,P}(-U_{k,2k-2j-3}, U_{k,2k-2j-2}), \end{aligned}$$

i.e. $f_{A,P}(\mathcal{Q}_{j+1}) \subseteq \text{ad}_{(X,P)}(\mathcal{Q}_j) = {}^cV_j$. $f_{A,P}$ injective implies $\dim({}^cV_j) \geq \dim(\mathcal{Q}_{j+1})$, $j = 0, 1, \dots, n-1$. Assume from now on that for any $j = 1, \dots, n-1$, $X^j \neq 0$; this condition is generically satisfied. Since $U_{j,0} = (-1)^j X^j$ we conclude $\text{ad}_{(X,P)}(0, U_{j,0}) = (0, 0)$ and hence $\dim(\mathcal{Q}_j) \geq 1 + \dim({}^cV_j)$ for $j \geq 1$. Hence we obtain

$$\dim(\mathcal{Q}_j) \geq 1 + \dim(\mathcal{Q}_{j+1}), \quad j = 1, \dots, n-1, \mathcal{Q}_n = 0. \quad (4.1)$$

Clearly $\mathcal{Q}_{n-1} = \text{span}(0, U_{n-1,0})$ so that $\dim(\mathcal{Q}_{n-1}) = 1$. Repeated application of (4.1) yields then $\dim(\mathcal{Q}_1) \geq n-1$, which combined with $n-1 \geq \dim({}^cV_0) \geq \dim(\mathcal{Q}_1)$ gives $\dim({}^cV) = n-1$.

THEOREM 4.3. Let $A = \text{diag}(a_1, \dots, a_n)$ have all entries distinct. The C. Neumann problem is a completely integrable Hamiltonian system, $n - 1$ generically independent integrals in involution being given by

$$f_{k+1}(X, P) = \frac{1}{2(k+1)} \text{Tr} \left[- \sum_{i=0}^k A^i X A^{k-i} + \sum_{\substack{i+j+l=k-1 \\ i, j, l \geq 0}} A^i P A^j P A^l \right].$$

REMARKS. (1) The geodesic spray on an ellipsoid in $\mathbf{R}^n$ with all axes distinct is also completely integrable and given by the Euler-Poisson equations on the *same* orbit

$$\dot{X} = [Q, X], \quad \dot{P} = [Q, P] + [X, A^{-1}],$$

for $Q_{ij} = -P_{ij}/a_i a_j$, with Hamiltonian $E(X, P) = -\frac{1}{2} \kappa(P, Q) + \kappa(X, A^{-1})$. It has the same integrals f_k since the previous equations can be written as

$$(-X + P\lambda + A\lambda^2)' = [-X + P\lambda + A\lambda^2, -Q - A^{-1}\lambda].$$

This follows easily from the work of Moser [10], [11] and has been independently observed by Adler and van Moerbeke [2] who also linearize the flow.

(2) The geodesic spray on S^{n-1} corresponds to $A = 0$ in the C. Neumann problem, or to $A = \text{Id}$ in the ellipsoidal problem. The Euler-Poisson equations on the *same* orbit are $\dot{X} = [P, X]$, $\dot{P} = 0$ and the integrals in involution are

$$f_k(X, P) = \begin{cases} \kappa \left(P^k - \left(\frac{1}{n} \text{Tr } P^k \right) \text{Id}, X \right), & k = \text{even}, \\ \frac{1}{2(k+1)} \text{Tr}(P^{k+1}), & k = \text{odd}. \end{cases}$$

The Hamiltonian is $-f_1(X, P) = -\frac{1}{2} \kappa(P, P)$. The prior proof of independence can be easily modified step-by-step to show that X_k , $k = 1, \dots, n - 1$, are generically independent.

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